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ItaCa

A fuzzy theory of types

Shreya Arya, Greta Coraglia, Paige North,
Sean O'Connor, Hans Riess, Ana Tenório

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Motivation: opinion dynamics

We need

- types and terms, because one opinion might have multiple proofs/reasons;
- fuzzy logic, because opinions are many-valued.

$$\Gamma \vdash r :_\alpha O$$

"Knowing Γ, I believe O because of r with confidence α."

Motivation: opinion dynamics

- types and terms
- fuzzy logic

	binary	*fuzzy*
propositions	$\{0, 1\}$	$[0, 1]$
types	$\mathbf{Set}$	$\Sigma_{S:\mathbf{Set}} S \to [0, 1]$

Confidence

What structure do we need on $[0, 1]$?

Definition

A commutative monoid $\mathbb{M} = (M, \cdot, 1)$ is

- *ordered* if there is a partial order $\leq$ on M such that $m \leq n$ implies $m \cdot x \leq n \cdot x$ for all $x \in M$;
- *unitally bounded* if $\leq$ has a top element and that is 1;
- *complete* if for each $m, n \in M$ there is $n^m \in M$ such that

$$x \leq n^m \quad \text{iff} \quad x \cdot m \leq n \qquad \text{for all } x \in M.$$

We call n^m the *internal hom* of m and n.

Commutative ordered monoids

- $\mathbb{2} = (\{0,1\}, \cdot, 1, \leq)$, with $\cdot$ and $\leq$ inherited by the usual ones on the reals
- $\mathbb{I} = ([0,1], \cdot, 1, \leq)$, as above
- $\mathbb{O}_X = (\mathcal{O}(X), \cap, X, \subseteq)$, where $\mathcal{O}(X)$ is the set of open subsets of X
- $\mathbb{L} = ([0,\infty], +, 0, \geq)$, with $+$ and $\geq$ inherited by the usual ones on the reals

more generally

- every commutative unital quantale
- every complete Heyting algebra (use $\wedge$ for $\cdot$)

These are actually all unitally bounded and complete, for example:

- ✓ in $\mathbb{I}$, $n^m = \min\{\frac{n}{m}, 1\}$ (thinking of the fraction in $[0,\infty]$ and defining $\frac{n}{0} = \infty$)
- ✓ in a quantale, $n^m = \bigvee_{x \cdot m \leq n} x$

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Fuzzy sets with values in $\mathbb{M}$

Definition

Call $\mathbf{Set}(\mathbb{M})$ the category having

- for objects $X = (X^o, |-|_X)$ where X^o is a set and $|-|_X$ is a function $X^o \to M$;
- morphisms $f : X \to Y$ are functions $f : X^o \to Y^o$ such that

$$|x|_X \leq |f(x)|_Y$$

for all $x \in X^o$.

✓ for ▯ we get sets with a *membership function*, we can interpret it to be

$$|x|_X = \alpha \quad \text{iff} \quad x \text{ is a member of } X \text{ with confidence } \alpha$$

	binary	*fuzzy*
propositions	$\{0, 1\}$	$[0, 1]$
types	$\mathbf{Set}$	$\Sigma_{S:\mathbf{Set}} S \to [0, 1]$

	binary	*fuzzy*
propositions	$\{0, 1\}$	$\mathbb{M}$
types	**Set**	$\mathbf{Set}(\mathbb{M})$

What do categories have to do with type theory?

$$\textit{Type theories} \rightleftarrows \mathbf{Set}\text{-}\textit{Categories}$$

$$\textit{Fuzzy type theories} \rightleftarrows \mathbf{Set}(\mathbb{M})\text{-}\textit{Categories}$$

Our strategy: enrich the categories, read the type theory!

Enriching categories: from Set to $\mathbf{Set}(\mathbb{M})$

Lemma

Both $\mathbb{M}$ and $\mathbf{Set}(\mathbb{M})$ support a monoidal structure.

For example, for $\mathbf{Set}(\mathbb{M})$:

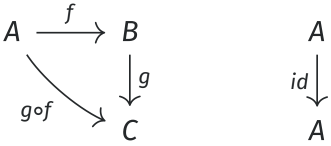
$$U \otimes V : \quad (U \otimes V)^{o} = U^{o} \times V^{o}, \; |(u,v)|_{U \otimes V} = |u|_U \cdot |v|_V$$

$$I : \quad (\{*\}, const_1)$$

Then we can use them as an enrichement:

- a $\mathbb{2}$-category has $(P \leq Q) = \underline{\hom}(P,Q) \in \{0,1\}$ hence propositions;
- a $\mathbb{I}$-category has $(P \leq_{\alpha} Q) = \underline{\hom}(P,Q) = \{\alpha\}$ hence "fuzzy propositions";
- a $\mathbb{L}$-category is a Lawvere metric space, $d(x,y) \in [0,\infty]$ and $d(x,y) + d(y,z) \geq d(x,z)$;
- a $\mathbf{Set}(\mathbb{M})$-category . . .

Composition vs monoidal product



$$\underline{\mathrm{hom}}(A, B) \otimes \underline{\mathrm{hom}}(B, C) \to \underline{\mathrm{hom}}(A, C), \qquad |f| \cdot |g| \leq |g \circ f|$$

$$I \to \underline{\mathrm{hom}}(A, A), \qquad 1 \leq |id|$$

Display-map categories

Definition (Taylor 1999, Hyland-Pitts 1987)

A *display-map category* is a pair $(\mathcal{C}, \mathcal{D})$ with $\mathcal{C}$ a category and $\mathcal{D} = \{p_A : \Gamma.A \to \Gamma\}$ a class of morphisms in $\mathcal{C}$ called *displays* or *projections* such that:

1. $\mathcal{C}$ has a terminal object 1;
2. for each $p_A : \Gamma.A \to \Gamma$ in $\mathcal{D}$ and $s : \Delta \to \Gamma$ in $\mathcal{C}$, there exists a choice of a pullback of p_A along s and it is again in $\mathcal{D}$,

$$\begin{array}{ccc} \Delta.A[s] & \xrightarrow{\bar{s}} & \Gamma.A \\ {\scriptstyle p_{A[s]}}\big\downarrow & & \big\downarrow{\scriptstyle p_A} \\ \Delta & \xrightarrow[s]{} & \Gamma \end{array}$$

3. $\mathcal{D}$ is closed under pre and post-composition with isomorphisms.

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3. $\mathcal{D}$ is closed under pre and post-composition with isomorphisms.

$\Gamma \vdash A \ \ \mathtt{type}$	$\Gamma.A \xrightarrow[p_A]{} \Gamma$
$\Gamma \vdash s : A$	$\Gamma.A \underset{p_A}{\overset{s}{\rightleftarrows}} \Gamma$
substitution	pullback along projections

Intuition

a **Set**($\mathbb{M}$)-category $\mathcal{C}$	an agent in the system
a context	a set of beliefs
a type (in context)	a belief (and its premises)
a term of type A	a proof of the belief A

- we want definite beliefs $\Rightarrow$ non-fuzzy types
- but their reasons might be subject to uncertainty $\Rightarrow$ fuzzy terms

Fuzzy display-map categories

Definition

A *fuzzy display-map category* is a pair $(\mathcal{C}, \mathcal{D})$ with $\mathcal{C}$ a $\mathbf{Set}(\mathbb{M})$-category and $\mathcal{D} = \{p_A : \Gamma.A \to \Gamma\}$ a class of morphisms in $\mathcal{C}$ called *fuzzy displays* or *fuzzy projections* such that:

1. $\mathcal{C}$ has a terminal object;
2. for each $p_A : \Gamma.A \to \Gamma$ in $\mathcal{D}$ and $s : \Delta \to \Gamma$ in $\mathcal{C}$, there exists a choice of a weighted pullback of p_A along s and its underlying map is again in $\mathcal{D}$,
3. $\mathcal{D}$ is closed under pre and post-composition with isomorphisms;
4. for all A, $|p_A|_{\underline{\mathrm{hom}}(\Gamma.A,\Gamma)} = 1$.

Projections and sections

Types are not fuzzy

For all A, $|\, p_A \,|_{\underline{\text{hom}}(\Gamma.A,\Gamma)} = 1$.

$$\Gamma \xrightarrow{\;s\;} \Gamma.A \xrightarrow{\;p_A\;} \Gamma \qquad (p_A \circ s = id)$$

Definition

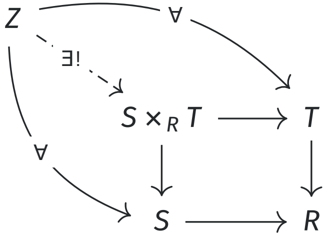
We say s is a α-*section* of p_A if s is a section of p_A and $|s| \geq \alpha$.

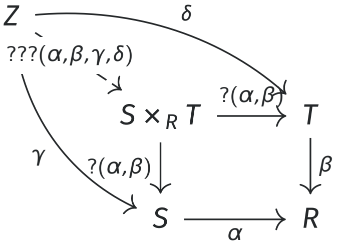
$$\Gamma \vdash s :_\alpha A \quad \text{and we have} \quad \frac{\Gamma \vdash s :_\alpha A}{\Gamma \vdash s :_\beta A} \quad \text{for all } \beta \leq \alpha$$

From now on, we just discuss the case of $\mathbb{M} = \mathbb{I}$.
Notice that all of the following results extend to the general case.

Substituting with uncertainty: weighted pullbacks

What is a pullback in **Set**(▯)?

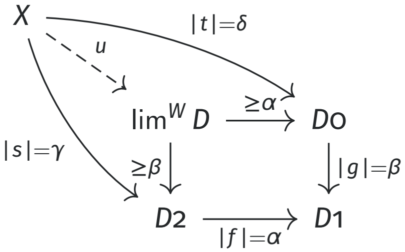




What do we ask of maps $S \leftarrow S \times_R T \rightarrow T$? *Here is where we pick weights.*
What happens to the map induced by the universal property of the pullback?

Substituting with uncertainty: weighted pullbacks

(Many calculations you don't want to see,
just know they involve this guy: $\hom_{\mathcal{C}}(X, \lim^W D) \cong \int_{\mathcal{D}} [W-, \underline{\hom}(X, D-)]$.)



$$|u| = \min\left(1, \frac{\gamma}{\beta}, \frac{\delta}{\alpha}\right)$$

Rules for fuzzy type theory

$$\frac{}{\vdash \diamond \ \mathtt{ctx}}\text{(C-Emp)} \qquad \frac{\Gamma \vdash A \ \mathtt{type}}{\vdash \Gamma, x : A \ \mathtt{ctx}}\text{(C-Ext)} \qquad \frac{\vdash \Gamma, x : A, \Delta \ \mathtt{ctx}}{\Gamma, x : A, \Delta \vdash x :_1 A}\text{(Var)}$$

$$\text{provided that } \beta \leq \alpha, \quad \frac{\Gamma \vdash t :_\alpha A}{\Gamma \vdash t :_\beta A}\text{(Cons)}$$

$$\frac{\Gamma, \Delta \vdash B \ \mathtt{type} \quad \Gamma \vdash A \ \mathtt{type}}{\Gamma, x : A, \Delta \vdash B \ \mathtt{type}}(\text{Weak}_{ty}) \qquad \frac{\Gamma, \Delta \vdash b :_\beta B \quad \Gamma \vdash A \ \mathtt{type}}{\Gamma, x : A, \Delta \vdash b :_\beta B}(\text{Weak}_{tm})$$

$$\frac{\Gamma, x : A, \Delta \vdash B \ \mathtt{type} \quad \Gamma \vdash a :_\alpha A}{\Gamma, \Delta[a/x] \vdash B[a/x] \ \mathtt{type}}(\text{Subst}_{ty}) \qquad \frac{\Gamma, x : A, \Delta \vdash b :_\beta B \quad \Gamma \vdash a :_\alpha A}{\Gamma, \Delta[a/x] \vdash b[a/x] :_\beta B[a/x]}(\text{Subst}_{tm})$$

Theorem

A fuzzy display-map category is sound and complete for the rules above.

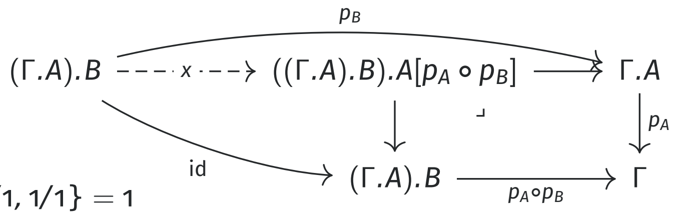
The variable rule

aka: here is where I'm pedantic

starting from $\vdash \Gamma, x : A, \Delta \ \mathtt{ctx}$ we want $\Gamma, x : A, \Delta \vdash \star :_? A$

(Assume $\Delta = y : B$ a single type, the general case works the same way.)

- Γ is a context
- A is a type in context Γ, hence there is a projection $p_A : \Gamma.A \to \Gamma$
- B is a type in context $\Gamma, x : A$, hence there is a projection $p_B : (\Gamma.A).B \to \Gamma.A$

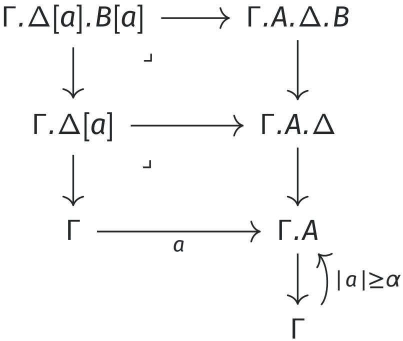


$|x| = \min\{1, 1/1, 1/1\} = 1$

$\Gamma, x : A, \Delta \vdash x :_1 A$ (actually, the second A is $A[p_A \circ p_B]$)

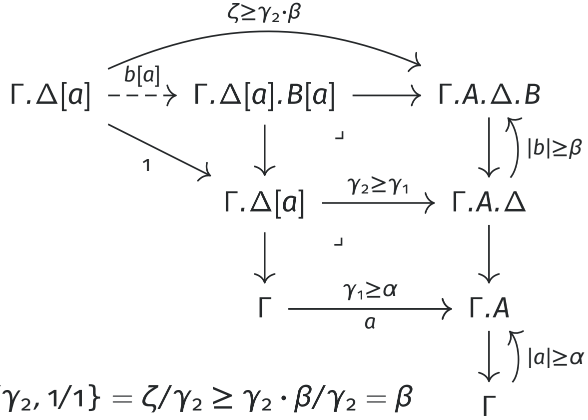
Substitution for types

starting from $\Gamma, x : A, \Delta \vdash B\ \mathtt{type}$ $\quad \Gamma \vdash a :_\alpha A$ we want $\Gamma, \Delta[a/x] \vdash B[a/x]$



Substitution for terms

starting from $\Gamma, x : A, \Delta \vdash b :_\beta B$ $\quad \Gamma \vdash a :_\alpha A$ $\quad$ we want $\Gamma, \Delta[a/x] \vdash b[a/x] :_? B[a/x]$



$$|b[a]| = \min\{1, \zeta/\gamma_2, 1/1\} = \zeta/\gamma_2 \geq \gamma_2 \cdot \beta/\gamma_2 = \beta$$

Opinions

a **Set**($\mathbb{M}$)-category $\mathcal{C}$	an agent in the system
a context	a set of beliefs
a type (in context)	a belief (and its premises)
a term of type A	a proof of the belief A
$1/^1\mathcal{C}$	tautologies
$E/^1\mathcal{C}$	facts induced by E
$E/^\alpha\mathcal{C}$	opinions induced by E with confidence α

Future work

- we have three possibilities to describe definitional equality
- study the behaviour of type constructors
- unpack more examples with different $\mathbb{M}$'s
- explore the dynamic side using **Set**($\mathbb{M}$)-valued sheaves (following Hansen-Ghirst 2020)

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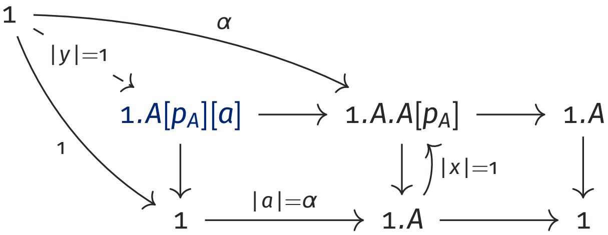
Something weird



$$\frac{\vdash a :_{\alpha} A}{\vdash y :_1 A[p_A][a]}$$

In the notation we have used so far, $y = x[a]$.
We have two types in the empty context, and they look very similar:

- a type A
- a type $A[p_A][a]$ obtained by extending A with itself, and then substituting a

but they are inherently different! How can we interpret this?
If I can prove A with confidence α,
I can prove (I can prove A with confidence α) with confidence 1.